A CRITERION FOR THE EQUIVALENCE OF FORMAL SINGULARITIES

By Konrad Möhring and Duco van Straten

Abstract. We prove a generalization of the finite determinacy theorem for isolated singularities. The maximal ideal occuring in the finite determinacy theorem is replaced by any ideal annihilating the first cotangent cohomology of a formal singularity over a Noetherian ring. An analogous result holds for finitely generated modules. As an application we give a criterion for the algebraizability of formal singularities and modules.

0. Introduction. In this paper we give a criterion for certain algebras over a noetherian ring S to be isomorphic, Theorem 1.1. Informally speaking, the criterion is the following stability assertion. Let the first cotangent cohomology $T^1(R/S)$ of $R = S[[x_1,\dots,x_n]]/I$ be annihilated by some power of an ideal $\mathfrak{a}$. Then any $S[[x_1,\dots,x_n]]/J$, such that generators of J and relations among the generators are congruent to generators and relations of I modulo a sufficiently high power of $\mathfrak{a}$, is right equivalent to R. If R is an isolated singularity over the field k, $T^1(R/k)$ is always annihilated by some power of the maximal ideal $(x_1,\dots,x_n)$, because the support of $T^1(R/k)$ is contained in the singular locus; so this generalizes known results on isolated singularities.

For a list of references on the subject, we refer to the introduction of [CS93]. Our proof is similar to Hironaka's proof of a criterion for the equivalence of isolated singularities sketched in [Hir69].

Just as for isolated singularities in Artin's paper [Art69, Th. 3.8], we deduce from our criterion the algebraizibility of a certain class of singularities, Theorem 1.3. This class includes the isolated singularities, generalizing Artin's result. Theorem 1.5 is the analogue of our main theorem for finitely generated modules over a field.

We will use the notation $P = S[[x_1,\dots,x_n]]$ throughout. We recall that the first cotangent cohomology $T^1(R/S)$ of an S-algebra $R = P/I$ is the cokernel of the natural map $Der_S(P,P) \longrightarrow Hom_P(I,R)$.

1. Results. Our main theorem is this:

Theorem 1.1. (Equivalence of singularities) *Let S be a noetherian commutative ring with* 1, *$P = S[[x_1,\dots,x_n]]$ and $\mathfrak{a} \subset P$ an ideal such that* $1-x$ *is invertible*

Manuscript received October 15, 2001; revised March 19, 2002.
American Journal of Mathematics 124 (2002), 1319–1327.

for all $x \in \mathfrak{a}$ *and* P *is* $\mathfrak{a}$*-complete, i.e.,* $(P, \mathfrak{a})$ *is a complete Zariski ring. Let* $I \subset P$ *be a proper ideal and write* $R := P/I$. *Assume that* $\mathfrak{a}^a T^1(R/S) = 0$ *for some* $a \in \mathbb{N}$. *For an exact sequence of P-modules*

$$P^r \xrightarrow{G} P^s \xrightarrow{F} P \to R \to 0,$$

there exist constants a_F, a_G *and* b, *such that the following holds: If* $c \in \mathbb{N}_0$, F' *and* G' *are matrices whose entries are congruent to those of* F *and* G *modulo* $\mathfrak{a}^{a_F+c}$ *and* $\mathfrak{a}^{a_G}$ *respectively, and if* $F' \circ G' = 0$, *then there is an automorphism* Φ *of* P *over* S *which is congruent to the identity modulo* $\mathfrak{a}^{a_F+c-b}$ *and carries the ideal* $I' := Im(F')$ *onto* $I = Im(F)$.

In particular, the theorem is valid if we choose $\mathfrak{a}$ to be the following ideal H_I, which can easily be computed from the given data.

Definition 1.2. Let $Jac(F)$ denote the jacobian matrix of partial derivatives of F. If A, B, C, D are subsets of indices, let G_{AB} and $Jac(F)_{CD}$ denote the corresponding submatrices.

We define the ideal $H_I \subset P$ to be generated by

$$\{det(G_{AB}) \cdot det(Jac(F)_{CD}) \mid \#A = \#B = p,\ \#C = \#D = s - p \text{ and } A \cup D = \{1, \dots, s\}\}.$$

The ideal H_I or rather $H_I + I$ describes the nonsmooth locus of R over S. Since the cotangent cohomology has support in the nonsmooth locus, a power of H_I annihilates T^1. Following Artin, [Art76, Part II], we outline a direct proof: Consider the complex

$$R^r \xrightarrow{G \otimes R} R^s \xrightarrow{Jac(F) \otimes R} R^n. \tag{1}$$

Localizing at a prime $\mathfrak{p} \supset I$ gives a split sequence iff $H_I \subset \mathfrak{p}$. In this case the dual complex of (1) is also a split sequence. In particular it is exact. Now $T^1(R/S)$ is the homology of this dual complex, so $T^1(R/S)$ is annihilated by some power of H_I.

The special case of Theorem 1.1 for an ideal defining the nonsmooth locus has already appeared, slightly modified, in [CS97, Th. 4.4]. However, our theorem is stronger, since the support of T^1 can be smaller than the nonsmooth locus, e.g. for rigid singularities.

Now we consider the special case that S is a field and $\mathfrak{a} = \mathfrak{m} = (x_1, \dots, x_n)$. Following Artins proof for isolated singularities [Art69, Th. 3.8], we deduce the algebraizability of singularities with $dim_k T^1(R/k) < \infty$.

THEOREM 1.3. *Let* k *be any field. Let* $I \subset \mathfrak{m} \subset P = k[[x_1, \dots, x_n]]$ *be an ideal,* $R := P/I$ *and* $dim_k T^1(R/k) < \infty$. *Let* $H = k\langle x_1, \dots, x_n \rangle$ *be the Henselization of the* *polynomial ring at the maximal ideal* $(x_1, \ldots, x_n)$*, i.e., the ring of algebraic power series.*

Then there is an ideal $J \subset H$ *and a formal automorphism* Φ *of* P*, which transforms the completion of* J *into* I*:*

$$\Phi(\hat{J}) = I.$$

Proof. The condition $dim_k T^1(R/k) < \infty$ is equivalent to $\mathfrak{m}^a T^1(R/k) = 0$ for some constant a. We choose a representation

$$P^r \xrightarrow{G} P^s \xrightarrow{F} P \to R \to 0$$

of R. So if $F = (f_i)$ and $G = (g_{ij})$, we have generators $f_1, \ldots, f_s$ of I and relations $\sum_i f_i g_{ij} = 0$. The f_i and g_{ij} are solutions of the following system of equations in the unknowns Y_i, Y_{ij}:

$$\sum_{i=1}^{s} Y_i Y_{ij} = 0, \qquad j = 1, \ldots, r.$$

Now we make use of the Artin approximation theorem as stated in [KPR75, Satz 5.2.1, (4)]:

THEOREM 1.4. (Artin Approximation Theorem) *Let* $H = k\langle x_1, \ldots, x_n \rangle$ *be the Henselization of the polynomial ring at the maximal ideal* $(x_1, \ldots, x_n)$*. We assume* $\bar{y}(x) \in P^N$ *to be a solution of a system of polynomial equations in* N *variables over* H*. Let* k *be any number. Then there is an algebraic solution* $y(x) \in H^N \subset P^N$*, approximating the given solution up to order* k*:*

$$\bar{y}(x) - y(x) \equiv 0 \mod \mathfrak{m}^k.$$

Choosing k to be bigger than the constants a_F and a_G in the theorem, we are done. □

By essentially the same proof as for Theorem 1.1 we obtain the following statement for finitely generated modules.

THEOREM 1.5. *Let* M *be a finitely generated module over* $P = k[[x_1, \ldots, x_n]]$ *with* $\mathfrak{a}^a Ext^1(M, M) = 0$*. Fix a representation*

$$P^r \xrightarrow{G} P^s \xrightarrow{F} P^t \to M \to 0$$

of M*, where* G *and* F *are matrices with entries in* P*. Then there are constants* a_F*,* a_G *and* b *such that the following holds: If* F' *and* G' *are matrices whose entries are congruent to those of* F *and* G *modulo* $\mathfrak{a}^{a_F + c}$ *and* $\mathfrak{a}^{a_G}$ *respectively,* $c \in \mathbb{N}_0$ *and if*

$F' \circ G' = 0$, then there is an automorphism of P^t which carries $Im(F')$ onto $Im(F)$. The automorphism is congruent to the identity modulo $\mathfrak{a}^{a_F+c-b}$.

COROLLARY 1.6. *A finitely generated module over $P = k[[x_1, \ldots, x_n]]$ with the property $\dim_k Ext^1(M,M) < \infty$ is algebraic, i.e., the completion of a module over the ring of algebraic power series $H = k\langle x_1, \ldots, x_n \rangle$.*

2. Proof of Theorem 1.1. We denote the entries of the matrices F and G by f_i and g_{ij} respectively. The exact sequence

$$P^r \xrightarrow{G} P^s \xrightarrow{F} I \to 0$$

gives us an embedding of the normal module $N = Hom_P(I, R)$ into R^s:

$$\begin{array}{ccccccc} 0 & \to & Hom_P(I,R) & \xrightarrow{F^*} & Hom_P(P^s,R) & \xrightarrow{\cong} & R^s, \\ & & n & \mapsto & F^*(n) & \mapsto & (n(f_1), \ldots, n(f_s)). \end{array}$$

The entries of F' and G' are

$$f'_i = f_i + \phi_i, \qquad \phi_i \in \mathfrak{a}^{a_F}, \tag{2}$$

$$g'_{ij} = g_{ij} + \gamma_{ij}, \qquad \gamma_{ij} \in \mathfrak{a}^{a_G}, \tag{3}$$

with $a_F, a_G \gg 0$. We will give explicit lower bounds for a_F and a_G later on in the proof. We have assumed that

$$\begin{aligned} 0 &= \sum f'_i g'_{ij} \\ &= \sum f_i g_{ij} + \sum \phi_i g_{ij} + \sum f_i \gamma_{ij} + \sum \phi_i \gamma_{ij}. \end{aligned}$$

The first summand is zero, the third is in the ideal $I = (f_1, \ldots, f_s)$ and the fourth is an element of $\mathfrak{a}^{a_F+a_G}$. So $\bar{n}(f_i) := (f'_i - f_i) = \phi_i$ defines a P-module homomorphism $\bar{n}\colon I \to P/(I + \mathfrak{a}^{a_F+a_G})$ with the property

$$\bar{n}(f_i) = \phi_i + (I + \mathfrak{a}^{a_F+a_G}).$$

We would like to find an element n in the normal module N of R, i.e., a homomorphism from I to $R = P/I$, that induces $\bar{n}$.

PROPOSITION 2.1. *Let P be any Noetherian ring, $\mathfrak{a} \subset P$ an ideal, and $\lambda\colon A \to B$ any homomorphism between finitely generated P-modules. Then there exists an integer $c = c(\lambda)$ with the following property: For all $x \in A$ and $p \in \mathbb{N}$ such that*

$$\lambda(x) \equiv 0 \mod \mathfrak{a}^{p+c} B$$

there exists an $\tilde{x} \in A$ *such that*

$$\begin{aligned} \lambda(\tilde{x}) &= 0 \\ \text{and} \quad \tilde{x} &\equiv x \mod \mathfrak{a}^p A. \end{aligned}$$

Proof. Consider the submodule $Im(\lambda) \subset B$. By the Artin-Rees lemma (cf. [Eis95]), there exists an integer c such that

$$Im(\lambda) \cap \mathfrak{a}^{p+c} B = \mathfrak{a}^p (Im(\lambda) \cap \mathfrak{a}^c B).$$

So if $\lambda(x) \in \mathfrak{a}^{p+c} B$ we must have $\lambda(x) = \sum_i r_i n_i$ with $r_i \in \mathfrak{a}^p$ and $n_i = \lambda(m_i) \in Im(\lambda)$. Then $\tilde{x} = x - \sum_i r_i m_i$ is just what we want. □

Now we apply this proposition to the P-modules $A = Hom_P(P^s, P)$ and $B = Hom_P(P^r, R)$ and the homomorphism

$$\begin{aligned} \lambda \colon \; A &\longrightarrow B, \\ \phi &\longmapsto \phi \circ G \mod I. \end{aligned}$$

Let's call the integer $c(\lambda)$ of the proposition c_1. Then we end up with $\tilde{\phi}_i$ such that

$$\tilde{\phi}_i \equiv \phi_i \mod \mathfrak{a}^{a_F + a_G - c_1} \tag{4}$$

with the property that

$$\sum \tilde{\phi}_i g_{ij} \equiv 0 \mod I.$$

Hence these $\tilde{\phi}_i$ describe an $n \in N = \mathrm{Hom}(I, R)$ defined by

$$n(f_i) = \tilde{\phi}_i + I. \tag{5}$$

Let's assume we have chosen $a_G > c_1$. As $\tilde{\phi}_i \equiv \phi_i \mod \mathfrak{a}^{a_F + a_G - c_1}$ by (4), this implies $\tilde{\phi}_i \equiv \phi_i \mod \mathfrak{a}^{a_F}$ and since $\phi_i \in \mathfrak{a}^{a_F}$ by (2) this leads to

$$\tilde{\phi}_i \in \mathfrak{a}^{a_F}. \tag{6}$$

We have embedded the normal module N into R^s by assigning to a homomorphism in N the s values on $f_1, \ldots, f_s$. So our n from (5) is mapped into $\mathfrak{a}^{a_F} R^s$. Applying Proposition 2.1 to the embedding $N \to R^s$, we obtain an integer c_2 depending only on the embedding, such that

$$n \in \mathfrak{a}^{a_F - c_2} N.$$

Next, we want to find a derivation $\theta \in Der_S(P,P)$, whose restriction to I induces n. The cokernel of the map from $Der_S(P,P)$ to N is by definition $T^1(R/S)$. We have assumed $\mathfrak{a}^a T^1(R/S) = 0$, so $\mathfrak{a}^a N$ is contained in the image of $Der_S(P,P)$ under this map. So n is induced by some

$$\theta \in \mathfrak{a}^{a_F - a - c_2} Der_S(P,P). \tag{7}$$

This means we have the equalities

$$n(f_i) = \theta(f_i) + I$$

and by (4) and (5) this implies

$$\theta(f_i) \equiv \phi_i \mod I + \mathfrak{a}^{a_F + a_G - c_1}. \tag{8}$$

But as by (7) $\theta \in \mathfrak{a}^{a_F - a - c_2} Der_S(P,P)$ and by (2) $\phi \in \mathfrak{a}^{a_F}$, we also know

$$\theta(f_i) \equiv \phi_i \mod \mathfrak{a}^{a_F - a - c_2}. \tag{9}$$

Applying the Artin-Rees lemma once more we find an integer c_3 such that

$$\mathfrak{a}^{p + c_3} \cap I = \mathfrak{a}^p(\mathfrak{a}^{c_3} \cap I) \subset \mathfrak{a}^p I. \tag{10}$$

We have chosen $a_G > c_1$. So $a_F - a - c_2 < a_F + a_G - c_1$ and (10) implies $\mathfrak{a}^{a_F - a - c_2} \cap (I + \mathfrak{a}^{a_F + a_G - c_1}) \subset \mathfrak{a}^{a_F - a - c_2 - c_3} I + \mathfrak{a}^{a_F + a_G - c_1}$. Combining this with (8) and (9) we get:

$$\theta(f_i) \equiv \phi_i \mod \mathfrak{a}^{a_F - a - c_2 - c_3} I + \mathfrak{a}^{a_F + a_G - c_1}. \tag{11}$$

We use the derivation θ to construct an automorphism Φ_{a_F} of $P = S[[x_1, \ldots, x_n]]$ by setting

$$\Phi_{a_F}(x_m) := x_m - \theta(x_m).$$

From (7) we deduce the two obvious inclusions

$$\theta(\mathfrak{a}^k) \subset \mathfrak{a}^{a_F - a - c_2 + k - 1} \tag{12}$$

$$\text{and} \quad \Phi_{a_F}(f) \equiv f - \theta(f) \mod \mathfrak{a}^{2(a_F - a - c_2)} \qquad \forall f \in P. \tag{13}$$

We first notice that

$$\Phi_{a_F} \equiv Id_P \mod \mathfrak{a}^{a_F - a - c_2}. \tag{14}$$

Further

$$\begin{aligned}
\Phi_{a_F}(f_i+\phi_i) &= \Phi_{a_F}(f_i)+\Phi_{a_F}(\phi_i) \\
&\equiv f_i+(\phi_i-\theta(f_i))-\theta(\phi_i) && \bmod \mathfrak{a}^{2(a_F-a-c_2)} \\
&\equiv f_i-\theta(\phi_i) && \bmod (\mathfrak{a}^{a_F-a-c_2-c_3}I+\mathfrak{a}^{a_F+a_G-c_1}) \\
&\equiv f_i && \bmod \mathfrak{a}^{2a_F-a-c_2-1}.
\end{aligned}$$

The first congruence follows from (13), the second from (11) and the third from (12). If we choose $a_G \geq c_1+1$ and $a_F \geq \max\{2a+2c_2+1, a+c_2+2, a_G+a+c_2+c_3\}$, we get

$$\begin{aligned}
\Phi_{a_F}(f_i+\phi_i) &\equiv f_i && \bmod \mathfrak{a}^{a_G}I+\mathfrak{a}^{a_F+1} \\
\Leftrightarrow \Phi_{a_F}(f_i+\phi_i) &= f_i+\psi_i+\phi_i'' && \text{with } \psi_i \in \mathfrak{a}^{a_G}I, \phi_i'' \in \mathfrak{a}^{a_F+1}.
\end{aligned}$$

Consider the vector $(f_i+\psi_i)$. It can be written as $(f_1,\ldots,f_s)\circ(1+\Psi_{a_F})$, where Ψ_{a_F} is a matrix with entries in $\mathfrak{a}^{a_G}$. Since P is $\mathfrak{a}$-complete, $1+\Psi_{a_F}$ is invertible and describes an automorphism of P^s. Set $\tilde{F} := F\circ(1+\Psi_{a_F})$ and $\tilde{G} := (1+\Psi_{a_F})^{-1}\circ G$. We get a new representation of R:

$$\begin{array}{ccccccccc}
P^r & \xrightarrow{\tilde{G}} & P^s & \xrightarrow{\tilde{F}} & P & \longrightarrow & R & \longrightarrow & 0 \\
& {\scriptstyle G}\searrow & \updownarrow{\scriptstyle \cong} & \nearrow{\scriptstyle F} & & & & & \\
& & P^s & & & & & &
\end{array}$$

We set $G'' = G'$ and $F'' = \Phi_{a_F}\circ F'$:

$$\begin{array}{ccccc}
P^r & \xrightarrow{G'=G''} & P^s & \xrightarrow{F'} & P \\
& & & {\scriptstyle F''}\searrow & \downarrow{\scriptstyle \Phi_{a_F}} \\
& & & & P\ .
\end{array}$$

Then $F''\circ G'' = 0$. We have shown that the entries of $\tilde{F}$ are congruent to those of F'' modulo $\mathfrak{a}^{a_F+1}$. The entries of $\tilde{G}$ are congruent to those of G modulo $\mathfrak{a}^{a_G}$, which in turn are congruent to those of $G''=G'$ modulo $\mathfrak{a}^{a_G}$, so the entries of $\tilde{G}$ are congruent to those of G'' modulo $\mathfrak{a}^{a_G}$.

So we have improved the situation by raising a_F by one. Now we want to use induction on a_F; to do this we have to check wether all those constants may be taken to be the same in the next step of our induction:

The constants a and c_3 only depend on $\mathfrak{a}$ and I.

The constant c_1 was found by applying Proposition 2.1 to the homomorphism

$$\begin{aligned}
\lambda:\ Hom_P(P^s,P) &\rightarrow Hom_P(P^r,R) \\
\phi &\mapsto \phi\circ G \mod I.
\end{aligned}$$

In the next step of our induction we will apply it to

$$\lambda' : \ \phi \mapsto \phi \circ (Id + \Psi_{a_F})^{-1} \circ G \quad \mathrm{mod}\, I.$$

That is to say: Instead of at λ we will be looking at the composition of λ with the automorphism $((Id+\Psi_{a_F})^{-1})^*$ of $Hom_P(P^s, P)$. Since the integer c_1 only depends on the image of λ, it will be the same as before.

The last constant we have to consider is c_2. It was found by applying the Artin-Rees lemma to the submodule $F^*(Hom(I,R)) \subset Hom_P(P^s,R) \cong R^s$. In the next step we will be considering the submodule $(1+\Psi_{a_F})^*(F^*(Hom(I,R)))$ in $Hom_P(P^s,R)$. But $(1+\Psi_{a_F})^*$ is a P-module automorphism of $Hom_P(P^s,R)$, so we can apply the following easy lemma:

LEMMA 2.2. *Let P be a ring, $\mathfrak{a} \subset P$ an ideal, $A \subset M$ two P-modules and $\varphi \in Aut_P(M)$. If*

$$A \cap \mathfrak{a}^{p+c} M = \mathfrak{a}^p (A \cap \mathfrak{a}^c M) \tag{15}$$

for some integers $p, c \in \mathbb{N}$, then

$$\varphi(A) \cap \mathfrak{a}^{p+c} M = \mathfrak{a}^p (\varphi(A) \cap \mathfrak{a}^c M). \tag{16}$$

In particular, if P is noetherian and M finitely generated, the Artin-Rees lemma gives rise to the same constants when applied to the two submodules A and $\varphi(A)$ of M.

Proof. It is trivial to see that for any two submodules B_1, B_2 of M we have $\varphi(B_1 \cap B_2) = \varphi(B_1) \cap \varphi(B_2)$ and also $\varphi(\mathfrak{a}^p B_1) = \mathfrak{a}^p \varphi(B_1)$. So the left resp. right side of (15) gets mapped to the left resp. right side of (16). □

Now let's do the induction. $\Phi_{a_F} \equiv Id \,\mathrm{mod}\, \mathfrak{a}^{a_F - a - c_2}$, so we have a limit $\Phi = \cdots \circ \Phi_{a_F+1} \circ \Phi_{a_F}$, which is an automorphism of P. In the same way we get a matrix Ψ with entries in some power of $\mathfrak{a}$ such that $(1+\Psi) = \prod (1 + \Psi_{a_F+k})$. By construction, $\Phi(f_i + \phi_i)$ is the ith component of $(f_1, \ldots, f_s) \circ (1+\Psi)$, so $\{f_i + \phi_i\}$ is being mapped to the generating system $\{f_i \circ (1+\Psi)\}$ of I, hence $\Phi(I') = I$. □

3. Proof of Theorem 1.5. The proof is the same as for Theorem 1.1. We will only check that the condition $\mathfrak{a}^a Ext^1(M,M) = 0$ for modules is the analogue to the condition $\mathfrak{a}^a T^1 = 0$ we had before. We fix a presentation

$$P^r \xrightarrow{G} P^s \xrightarrow{F} P^t \to M \to 0$$

of M and consider a perturbation

$$P^r \xrightarrow{G+\Gamma} P^s \xrightarrow{F+\Phi} P^t$$

which is an exact sequence. Then the $(t \times s)$-matrix Φ defines a homomorphism $Im(F) \cong P^s/(Im(G) \xrightarrow{\Phi} (P^t/Im(F))/\mathfrak{a}^{\gg}$. We approximate this homomorphism by a homomorphism to $P^t/Im(F)$. Now the crucial point is to extend this homomorphism from $Im(F)$ to all of P^t. We begin with the exact sequence

$$0 \to Im(F) \to P^t \to M \to 0.$$

This gives us a long exact sequence which starts like this:

$$0 \to Hom(M,M) \to Hom(P^t,M) \to Hom(Im(F),M) \to Ext^1(M,M) \to \cdots.$$

So if $\mathfrak{a}^a Ext^1(M,M) = 0$, all homomorphisms in $\mathfrak{a}^a Hom(Im(F),M)$ can be extended to P^t. Finally we lift this extension from $Hom(P^t,M)$ to an automorphism $\Psi \in Hom(P^t,P^t)$. The automorphism $Id_{P^t} + \Psi$ is the analogue to the automorphism we have constructed above. The rest of the proof is exactly as for Theorem 1.1. It consists mainly of keeping track of the powers of $\mathfrak{a}$ up to which things vanish. We leave the details to the reader.

JOHANNES-GUTENBERG UNIVERSITÄT, FB17 – MATHEMATIK, 55099 MAINZ, GERMANY
E-mail: konrad@mathematik.uni-mainz.de
E-mail: straten@mathematik.uni-mainz.de

REFERENCES

[Art69] M. Artin, Algebraic approximation of structures over complete local rings, *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 23–58.

[Art76] ———, *Lectures on Deformations of Singularities*, Tata Institute of Fundamental Research, Bombay, 1976.

[CS93] S. D. Cutkosky and H. Srinivasan, An intrinsic criterion for isomorphism of singularities, *Amer. J. Math* **115** (1993), 789–821.

[CS97] ———, Equivalence and finite determinancy of mappings, *J. Algebra* **188** (1997), 16–57.

[Eis95] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry, Grad. Texts in Math.*, vol. 150, Springer-Verlag, New York, 1995.

[Hir69] H. Hironaka, Formal line bundles along exceptional loci, *Algebraic Geometry, Bombay Colloq. 1968*, Oxford Univ. Press, London, 1969, pp. 201–218.

[KPR75] H. Kurke, G. Pfister, and M. Roczen, *Henselsche Ringe*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975.